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AFDELING ZUIVERE WISKUNDE ZW 31/74 OCTOBER

J. VAN DE LUNE

ON THE ASYMPTOTIC BEHAVIOUR OF A SEQUENCE ARISING IN COMPUTER SCIENCE

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

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AMS (MOS) subject classification scheme (1970): 40A05 10H05 68A50

On the asymptotic behaviour of a sequence arising in Computer Science

by

J. van de Lune

ABSTRACT

This report mainly deals with the asymptotic behaviour of the sequence $\{\phi_x(n)\}_{n=1}^{\infty}$ where $\phi_x(n) = n \cdot \sum_{k=1}^{\infty} x^k(1-x^k)^{n-1}$, $(0<x<1;\ n \in \mathbb{N})$, which plays a certain role in the theory of access structures in computer science.

In relation to some generalizations of $\phi_x(n)$ we also discuss a conjecture concerning the ordinates of the non-trivial zeros of Riemann's zeta function.

KEYWORDS & PHRASES: *Computer Science, access structure, search depth, Riemann-zeta function, sequence.*

0. INTRODUCTION

The origin of this note lies in a request to determine the limit of the sequence $\{\phi_x(n)\}_{n=1}^{\infty}$ where

$$(1) \qquad \phi_x(n) = n \cdot \sum_{k=1}^{\infty} x^k(1-x^k)^{n-1}, \quad (0<x<1;\ n \in \mathbb{N})$$

This sequence plays a certain role in computer science as will be described briefly by L.G.L.T. MEERTENS, the proposer of the problem:

□ For a given set E (for example identifiers) one wants to construct an "access structure". To every element $e \in E$ one assigns an infinite sequence of keys $a_0a_1a_2\ldots$, $0 \le a_i < m$, m and a_i integers such that every sequence of keys uniquely determines its corresponding element. Let V be the set of all pairs $(e,a_0a_1a_2\ldots)$. The access structure S for V is recursively defined by:

If $V = \emptyset$ then S is "empty" (one memory word).

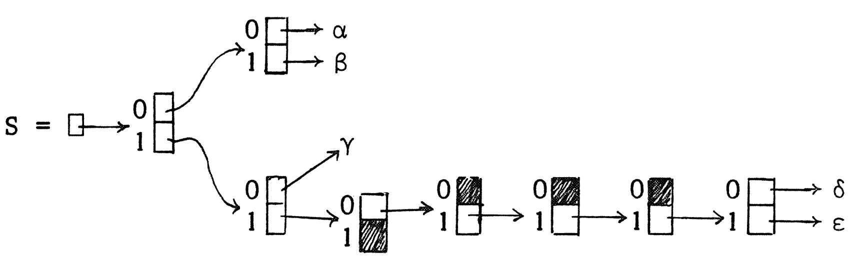
If $|V| = 1$ then S is a reference to the data corresponding to the single element of V.

If $|V| > 1$ then S is a reference to a row $S_0,S_1,\ldots,S_{m-1}$ where S_i is an access structure for the set $V_i = \{(e,a_1a_2\ldots) \mid (e,ia_1a_2\ldots) \in V\}$.

Example: Take $m=2$ and let V be the 5 element set

$$V = \{(\alpha,0000110100\ldots),\ (\beta,0101010110\ldots),\ (\gamma,1011101100\ldots),$$
$$(\delta,1101110000\ldots),\ (\varepsilon,1101111100\ldots)\}.$$

Then S may be pictured as follows



If the n element set V has been obtained by choosing the keys a_i at random and independently from the set $\{0,1,2,\ldots,m-1\}$ then β_n, the expected memory space occupied by the access structure for V, is determined by the relations:

(2)
$$\beta_0 = \beta_1 = 1$$
$$\beta_n = 1 + m \sum_{k=0}^{n} \binom{n}{k} \frac{1}{m^k} (1 - \frac{1}{m})^{n-k} \beta_k, \quad (n \geqq 2).$$

Defining γ_n by $\gamma_n = \frac{\beta_n - 1}{m}$ we get

(3)
$$\gamma_0 = \gamma_1 = 0$$
$$\gamma_n = 1 + m \sum_{k=0}^{n} \binom{n}{k} \frac{1}{m^k} (1 - \frac{1}{m})^{n-k} \gamma_k, \quad (n \geqq 2).$$

We may interpret γ_n as the expected number of rows of (subordinate) access structures. Every such row is uniquely determined by an initial sequence of keys $a_0 a_1 \ldots a_{k-1}$, $k \geqq 1$. It is possible to determine γ_n by summing (over all initial sequences) the probabilities that $V_{a_0 a_1 \ldots a_{k-1}}$ contains at least two elements. This yields

(4)
$$\gamma_n = \sum_{k=1}^{\infty} m^k \left\{1 - (1 - \frac{1}{m^k})^n - \frac{n}{m^k} (1 - \frac{1}{m^k})^{n-1}\right\}.$$

In a similar manner one finds that the expected search depth is

(5)
$$\delta_n = \sum_{k=0}^{\infty} \left\{1 - (1 - \frac{1}{m^k})^n - \frac{n}{m^k} (1 - \frac{1}{m^k})^{n-1}\right\}.$$

It seems interesting to investigate the asymptotic behaviour of the sequences γ_n and δ_n. □

In order to study γ_n we consider

(6)
$$\gamma_{n+1} - \gamma_n = n \sum_{k=1}^{\infty} \frac{1}{m^k} (1 - \frac{1}{m^k})^{n-1} = \phi_{\frac{1}{m}}(n).$$

One would expect that $\phi_{\frac{1}{m}}(n)$, for large values of n, has approximately the same size as

$$(7) \qquad n \int_0^\infty \frac{1}{m^u} \left(1 - \frac{1}{m^u}\right)^{n-1} du = \frac{1}{\log m},$$

so that one is tempted to conjecture that

$$\lim_{n\to\infty} \frac{\gamma_n}{n} = \frac{1}{\log m}.$$

However, after a more detailed investigation of the asymptotic behaviour of $\phi_x(n)$ for large values of n (throughout this note x will be an arbitrary but *constant* number in the open interval (0,1)), it was discovered that $\phi_x(n)$, contrary to all expectations, does *not* have a limit for any x. Moreover, it turned out that $\phi_x(n)$ does *not* even have an Abel-limit so that this sequence does *not* have a Cesàro-limit (of any order) either. In particular it follows that $\frac{\gamma_n}{n}$ does not have a limit when $n \to \infty$ for any $m \in \mathbb{N}$. We will show, however, that $\phi_x(n)$ (and hence $\frac{\gamma_n}{n}$) oscillates between positive bounds (depending on x). From these results one easily derives that the asymptotic behaviour of the search depth δ_n is given by $\delta_n \sim \frac{\log n}{\log m}$, $(n\to\infty)$.

Furthermore, we will discuss some analogues and analytical generalizations of the above problem. Finally we consider a sequence (or rather its continuous analogue) similar to $\phi_x(n)$, the study of which leads quite naturally to an (unsolved) problem concerning the ordinates of the nontrivial zeros of Riemann's ζ-function.

1. THEOREM 1.1. *For every fixed* $x \in (0,1)$ *the sequence* $\{\phi_x(n)\}_{n=1}^\infty$ *does not have an Abel-limit.*

PROOF. In order to prove this we proceed by contradiction, assuming that $\phi_x(n)$ has an Abel-limit (= ϕ_x, say). Since $0 < \phi_x(n) \leq n \sum_{k=1}^\infty x^k = \frac{nx}{1-x}$ for all $n \in \mathbb{N}$ it is clear that the power series

$$(8) \qquad \sum_{n=1}^\infty \phi_x(n)\, z^{n-1}$$

converges for $|z| < 1$.

Observe that for $|z| < 1$ we have

$$\text{(9)} \qquad \sum_{n=1}^{\infty} \phi_x(n)\, z^{n-1} = \sum_{n=1}^{\infty} \left\{ n \sum_{k=1}^{\infty} x^k (1-x^k)^{n-1} \right\} z^{n-1} =$$

$$= \sum_{k=1}^{\infty} x^k \sum_{n=1}^{\infty} n(1-x^k)^{n-1} z^{n-1} = \sum_{k=1}^{\infty} \frac{x^k}{(1-(1-x^k)z)^2} .$$

Hence, our assumption that $\phi_x(n)$ has an Abel-limit is seen to be equivalent to

$$\text{(10)} \qquad \lim_{z\uparrow 1}(1-z) \cdot \sum_{k=1}^{\infty} \frac{x^k}{(1-(1-x^k)z)^2} = \phi_x .$$

Writing $z = 1-t$, $0 < t < 1$, we obtain

$$\text{(11)} \qquad \phi_x = \lim_{t\downarrow 0} \sum_{k=1}^{\infty} \frac{x^k t}{(x^k+t(1-x^k))^2} = \lim_{t\downarrow 0} \sum_{k=1}^{\infty} \frac{tx^{-k}}{(1+t(x^{-k}-1))^2} .$$

Now let t run through the numbers x^{r+u} where u is some fixed number in $[0,1)$ whereas r runs through $\mathbb{N}$. Note that then

$$\text{(12)} \qquad \sum_{k=1}^{\infty} \frac{tx^{-k}}{(1+t(x^{-k}-1))^2} = \sum_{k=1}^{\infty} \frac{x^{-k+r+u}}{(1+x^{-k+r+u}-x^{r+u})^2} = \sum_{k<r} \frac{x^{k+u}}{(1+x^{k+u}-x^{r+u})^2} =$$

$$= \sum_{k<r} \frac{x^{k+u}}{(1+x^{k+u})^2 \left(1-\frac{x^{r+u}}{1+x^{k+u}}\right)^2} < \sum_{k<r} \frac{x^{k+u}}{(1+x^{k+u})^2(1-x^{r+u})^2} <$$

$$< (1-x^{r+u})^{-2} \cdot \sum_{k=-\infty}^{\infty} \frac{x^{k+u}}{(1+x^{k+u})^2} .$$

Letting r tend to infinity we find that

$$\text{(13)} \qquad \phi_x \leqq \sum_{k=-\infty}^{\infty} \frac{x^{k+u}}{(1+x^{k+u})^2} .$$

On the other hand we have

$$(14)\qquad \sum_{k=1}^{\infty} \frac{tx^{-k}}{(1+t(x^{-k}-1))^2} = \sum_{k<r} \frac{x^{k+u}}{(1+x^{k+u}-x^{r+u})^2} > \sum_{k=-N}^{N} \frac{x^{k+u}}{(1+x^{k+u}-x^{r+u})^2}$$

for all $r > N$.

Keeping N fixed and letting r tend to infinity we obtain

$$(15)\qquad \phi_x \geqq \sum_{k=-N}^{N} \frac{x^{k+u}}{(1+x^{k+u})^2}\,.$$

Since N may be chosen as large as we please it follows that

$$(16)\qquad \phi_x \geqq \sum_{k=-\infty}^{\infty} \frac{x^{k+u}}{(1+x^{k+u})^2}\,.$$

Hence

$$(17)\qquad \phi_x = \sum_{k=-\infty}^{\infty} \frac{x^{k+u}}{(1+x^{k+u})^2}\,.$$

Since ϕ_x clearly does not depend on u we find that the (indeed convergent) series

$$(18)\qquad \sum_{k=-\infty}^{\infty} \frac{x^{k+u}}{(1+x^{k+u})^2}\,,$$

(representing a periodic function on $\mathbb{R}$ with period 1) is *constant* as a function of u. Hence

$$(19)\qquad \phi_x = \sum_{k=-\infty}^{\infty} \frac{x^{k+u}}{(1+x^{k+u})^2} = \sum_{k=-\infty}^{\infty} \frac{x^{-k-u}}{(1+x^{-k-u})^2}\,,\ \forall u \in \mathbb{R}.$$

Writing $\frac{1}{x} = e^{\alpha}$, $(\alpha>0)$, and replacing u by $\frac{u}{2\pi}$ we obtain

$$(20)\qquad \phi_x = \sum_{k=-\infty}^{\infty} \frac{e^{\alpha(k+\frac{u}{2\pi})}}{(1+e^{\alpha(k+\frac{u}{2\pi})})^2}\,.$$

The last series represents a continuously differentiable 2π periodic function of u on $\mathbb{R}$ and may thus be represented pointwise by its Fourier series. Since this function is constant, all except one of its Fourier coefficients must vanish. We compute these vanishing Fourier coefficients a_ν, $\nu \in \mathbb{Z}\setminus\{0\}$, as follows

(21) $$a_\nu = \frac{1}{2\pi}\int_0^{2\pi} e^{\nu u i} \sum_{k=-\infty}^{\infty} \frac{e^{\alpha(k+\frac{u}{2\pi})}}{(1+e^{\alpha(k+\frac{u}{2\pi})})^2}\, du =$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_0^{2\pi} \frac{e^{\nu u i+\alpha(k+\frac{u}{2\pi})}}{(1+e^{\alpha(k+\frac{u}{2\pi})})^2}\, du = \sum_{k=-\infty}^{\infty}\int_k^{k+1} \frac{e^{\nu 2\pi(t-k)i+\alpha t}}{(1+e^{\alpha t})^2}\, dt =$$

$$= \int_{-\infty}^{\infty} \frac{e^{\nu 2\pi t i+\alpha t}}{(1+e^{\alpha t})^2}\, dt = \frac{1}{\alpha}\int_0^{\infty} \frac{u^w}{(1+u)^2}\, du, \text{ where } w = \frac{\nu 2\pi i}{\alpha}.$$

Hence

$$a_\nu = \frac{1}{\alpha}\,\Gamma(1-w)\Gamma(1+w).$$

Since the Γ-function does not have any zeros at all it follows that $a_\nu \neq 0$ for all $\nu \in \mathbb{Z}$. Hence the assumption that $\phi_x(n)$ has an Abel-limit leads to a contradiction, proving the theorem. □

<u>THEOREM 1.2.</u>

(22.1) $$\limsup_{n\to\infty} \phi_x(n) \leqq \frac{1}{x}\sum_{k=-\infty}^{\infty} x^k e^{-x^k},$$

(22.2) $$\liminf_{n\to\infty} \phi_x(n) \geqq x \sum_{k=-\infty}^{\infty} x^k e^{-x^k}.$$

PROOF. Define

$$(23)\qquad r_n = \left[-\frac{\log n}{\log x}\right] \text{ and } u_n = \left\{-\frac{\log n}{\log x}\right\} \overset{\text{def}}{=} -\frac{\log n}{\log x} - r_n.$$

Then $n = x^{-r_n - u_n}$ with r_n a nonnegative integer and $0 \leqq u_n < 1$. Now observe that $\lim_{n\to\infty} \Sigma_{k=1}^{\infty} x^k(1-x^k)^n = 0$, so that

$$(24)\qquad \phi_x(n+1) = n \sum_{k=1}^{\infty} x^k(1-x^k)^n + o(1) =$$

$$= \sum_{k=1}^{\infty} x^{k-r_n-u_n}\left(1 - \frac{x^{k-r_n-u_n}}{n}\right)^n + o(1) = \sum_{k>-r_n} x^{k-u_n}\left(1 - \frac{x^{k-u_n}}{n}\right)^n + o(1).$$

Now observe that for $k>-r_n$ one has

$$(25)\qquad 0 < \frac{x^{k-u_n}}{n} < 1$$

so that

$$(26)\qquad -\log\left(1 - \frac{x^{k-u_n}}{n}\right) = \sum_{m=1}^{\infty} \frac{x^{m(k-u_n)}}{m\cdot n^m} > \frac{x^{k-u_n}}{n}$$

which may be rewritten as

$$(27)\qquad \left(1 - \frac{x^{k-u_n}}{n}\right)^n < e^{-x^{k-u_n}}.$$

Hence

$$(28)\qquad \phi_x(n+1) + o(1) < x^{-u_n} \sum_{k>-r_n} x^k e^{-x^{k-u_n}} <$$

$$< \frac{1}{x} \sum_{k>-r_n}^{\infty} x^k e^{-x^k} < \frac{1}{x} \sum_{k=-\infty}^{\infty} x^k e^{-x^k},$$

and (22.1) follows. On the other hand, if n is large enough then $r_n > N$ so that

$$(29)\qquad \phi_x(n+1) + O(1) > \sum_{k=-N}^{N} x^{k-u_n}\left(1 - \frac{x^{k-u_n}}{n}\right)^n \geqq \sum_{k=-N}^{N} x^k\left(1 - \frac{x^{k-1}}{n}\right)^n .$$

Hence

$$(30)\qquad \liminf_{n\to\infty} \phi_x(n) \geqq \sum_{k=-N}^{N} x^k e^{-x^{k-1}}, \quad \forall N \in \mathbb{N},$$

and (22.2) follows.

<u>REMARK</u>. The continuous analogue $\psi_x(t)$ of $\phi_x(t)$, where

$$(31)\qquad \psi_x(t) = t\int_0^\infty x^u(1-x^u)^{t-1}du, \quad (0<x<1;\ t>0)$$

behaves much simpler than $\phi_x(t)$ because a simple change of variable ($x^u = v$) shows that $\psi_x(t)$ is actually constant as a function of t,

$$(32)\qquad \psi_x(t) = \frac{1}{\log \frac{1}{x}} .$$

Returning to $\phi_x(n)$, we saw that the crucial step in the proof of theorem 1.1 was to show that for *no* $\alpha > 0$ the periodic function

$$(33)\qquad \sum_{k=-\infty}^{\infty} \frac{e^{\alpha(k+u)}}{(1+e^{\alpha(k+u)})^2}, \quad (u\in\mathbb{R})$$

is actually constant (as a function of u). In view of some problems to appear later on in this note it is of some importance to prove this kind of thing "directly" instead of using Fourier expansions. We may, for example, proceed as follows. Assume

$$(34)\qquad \sum_{k=-\infty}^{\infty} \frac{e^{\alpha(k+u)}}{(1+e^{\alpha(k+u)})^2} = C = C_\alpha, \ \forall u \in \mathbb{R},$$

where C is some positive constant (depending on α). Writing $e^{\alpha u} = s$ we have

(35) $$\frac{C}{s} = \sum_{k=-\infty}^{\infty} \frac{e^{\alpha k}}{(1+se^{\alpha k})^2}, \quad \forall s > 0.$$

However, the last series represents an analytic function of the complex variable s with singularities in the points $s=0$ and $s = -e^{-\alpha k}$, $k \in \mathbb{Z}$. Since this is *not* in agreement with the analytic behaviour of Cs^{-1} we arrive at a contradiction.

As another generalization of $\phi_x(n)$ we consider the function

(36) $$\mu_x(t) = t^{\sigma} \cdot \sum_{k=1}^{\infty} x^{\alpha k}(1-x^{\beta k})^{\delta t}, \quad (0<x<1;\ t>0)$$

where α,β and δ are positive constants whereas σ denotes the fraction $\frac{\alpha}{\beta}$.

We will show that $\mu_x(t)$ does not tend to a limit (when t tends to infinity) for any choice of the positive parameters x,α,β and δ. Again we proceed by contradiction and assume that $\lim_{t\to\infty} \mu_x(t)$ exists (and $= \mu_x$, say). Letting t run through the points $t = x^{-\beta(r+u)}$, where $u \in [0,1)$ is fixed and $r \in \mathbb{N}$, we may show in a similar manner as before that

(37) $$\mu_x = \sum_{n=-\infty}^{\infty} x^{\alpha(n-u)} e^{-\delta x^{\beta(n-u)}}$$

or

(38) $$\mu_x = \sum_{n=-\infty}^{\infty} x^{-\alpha(n+u)} e^{-\delta x^{-\beta(n+u)}}.$$

Changing the notation slightly we thus find that there are positive constants α,β and δ such that

(39) $$\sum_{n=-\infty}^{\infty} e^{\alpha(n+u)} e^{-\delta e^{\beta(n+u)}} = C, \quad \forall u \in \mathbb{R},$$

for some positive constant $C = C(\alpha,\beta,\delta)$.

Writing $e^{\beta u} = s$ we obtain

(40) $$C = \sum_{k=-\infty}^{\infty} e^{\alpha k}(e^{\beta u})^{\frac{\alpha}{\beta}} e^{-\delta e^{\beta k} e^{\beta u}} = \sum_{k=-\infty}^{\infty} e^{\alpha k} s^{\frac{\alpha}{\beta}} e^{-\delta e^{\beta k} s}, \quad \forall s > 0,$$

or equivalently (putting $\frac{\alpha}{\beta} = \sigma$)

(41) $$Cs^{-\sigma} = \sum_{k=-\infty}^{\infty} e^{\alpha k} e^{-\delta s e^{\beta t}}, \quad \forall s > 0.$$

Now differentiate both sides r times in order to obtain

(42) $$C(-\sigma)(-\sigma-1)\dots(-\sigma-r+1)s^{-\sigma-r} = \sum_{k=-\infty}^{\infty} e^{\alpha k}(-1)^r \delta^r e^{r\beta k} e^{-\delta s e^{\beta k}}, \quad \forall s > 0.$$

This may be rewritten as

(43) $$C\sigma(\sigma+1)\dots(\sigma+r-1)s^{-\sigma-r} = \sum_{k=-\infty}^{\infty} e^{\alpha k} \delta^r e^{r\beta k} e^{-\delta s e^{\beta k}}, \quad \forall s > 0.$$

Since all terms of the last series are positive, we find (only taking the term corresponding to $k = 0$)

(44) $$C\sigma(\sigma+1)\dots(\sigma+r-1)s^{-\sigma-r} > \delta^r e^{-\delta s}, \quad \forall s > 0.$$

Now choose $s = \frac{r+\sigma}{\delta}$. Then we have

(45) $$C\sigma(\sigma+1)\dots(\sigma+r-1)\left(\frac{\delta}{r+\sigma}\right)^{\sigma+r} = C\frac{\sigma(\sigma+1)\dots(\sigma+r-1)}{(r-1)!(r-1)^{\sigma}} \frac{r!}{r}(r-1)^{\sigma} \cdot \frac{\delta^{\sigma+r}}{(r+\sigma)^{\sigma+r}} =$$

$$= C\frac{\sigma(\sigma+1)\dots(\sigma+r-1)}{(r-1)!(r-1)^{\sigma}} \frac{r^r e^{-r}\sqrt{2\pi r}}{r} \frac{(r-1)^{\sigma} r!}{r^r e^{-r}\sqrt{2\pi r}} \frac{\delta^{\sigma+r}}{(r+\sigma)^{\sigma+r}} =$$

$$= C\frac{\sigma(\sigma+1)\dots(\sigma+r-1)}{(r-1)!(r-1)^{\sigma}} \left(\frac{r}{r+\sigma}\right)^r e^{-r} \frac{\sqrt{2\pi}}{\sqrt{r}} \frac{(r-1)^{\sigma}}{(r+\sigma)^{\sigma}} \cdot \frac{r!}{r^r e^{-r}\sqrt{2\pi r}} \cdot \delta^{\sigma+r} >$$

$$> \delta^r e^{-r-\sigma}.$$

Now cancel the factors e^{-r} and δ^r and let r tend to infinity, yielding

(46) $$C \frac{1}{\Gamma(\sigma)} e^{-\sigma}.0.1.1.\delta^{\sigma} \geqq e^{-\sigma},$$

which is a palpable contradiction. □

REMARK. Another way of proving that the function in (39) can not be constant in u is to show that none of its Fourier coefficients vanish. The detailed computations which are straightforward are left to the reader.

Also, in case $\sigma < 1$, the complex function argument used before is applicable here. In (39) replace $e^{\beta u}$ by v. This yields

(47) $$Cv^{-\sigma} = \sum_{n=-\infty}^{\infty} e^{\alpha n} e^{-\delta v e^{\beta n}}, \quad (v>0).$$

Now take Laplace transforms in order to obtain

(48) $$Cs^{\sigma-1}\Gamma(1-\sigma) = \sum_{n=-\infty}^{\infty} \frac{e^{\alpha n}}{s+\delta e^{\beta n}}, \quad (s>0).$$

Since the analytic continuation of the left and right hand side do not agree, we arrive at a contradiction.

2. In this section we will consider a related problem in which we will encounter some real difficulties. We will investigate the question whether there exist any positive constants α and β such that the function

(49) $$\theta_x(t) = t^{\frac{\alpha}{\beta}} \sum_{k=1}^{\infty} \frac{x^{\alpha k}}{1+(1+x^{\beta k})^t}, \quad (t>0)$$

has a limit when t tends to infinity. Suppose $\lim_{t\to\infty} \theta_x(t)$ exists (and $= \theta_x$, say). Fix some $u \in [0,1)$ and let t run through the numbers $t = t_{r,u} = x^{-\beta(r+u)}$ where $r \in \mathbb{N}$. Then we have

(50) $$\theta_x(t) = \sum_{k=1}^{\infty} \frac{x^{\alpha(k-r-u)}}{1+(1+\frac{x^{\beta(k-r-u)}}{t})^t} = \sum_{n>-r} \frac{x^{\alpha(n-u)}}{1+(1+\frac{x^{\beta(n-u)}}{t})^t} >$$

$$> \sum_{n=-N}^{N} \frac{x^{\alpha(n-u)}}{1+(1+\frac{x^{\beta(n-u)}}{t})^t} .$$

As before it easily follows that

(51) $$\theta_x \geqq \sum_{n=-\infty}^{\infty} \frac{x^{\alpha(n-u)}}{1+e^{x^{\beta(n-u)}}} .$$

On the other hand we have for $r > N$

(52) $$\theta_x(t) = \{ \sum_{-r<n<-N} + \sum_{n=-N}^{N} + \sum_{n>N} \} \frac{x^{\alpha(n-u)}}{1+(1+\frac{x^{\beta(n-u)}}{t})^t} .$$

Let a be the smallest natural number such that $a\beta > \alpha$. Observe that for $t \geqq a$ and $0 < z < 1$

(53) $$(1+z)^t = 1 + \frac{z}{1!} t + \frac{z^2}{2!} t(t-1) + \ldots + \frac{z^a}{a!} t(t-1)\ldots(t-a+1)(1+vz)^{t-a}$$

for some $0 < v < 1$.

so that

(54) $$(1+\frac{x^{\beta(n-u)}}{t})^t > \frac{x^{a\beta(n-u)}}{t^a a!} t(t-1)\ldots(t-a+1).$$

It follows that for $t \geqq a$

(55) $$\sum_{-r<n<-N} \frac{x^{\alpha(n-u)}}{1+(1+\frac{x^{\beta(n-u)}}{t})^t} < \sum_{-r<n<-N} \frac{x^{\alpha(n-u)}}{\frac{x^{a\beta(n-u)}}{t^a a!} t(t-1)\ldots(t-a+1)} =$$

$$= a! \frac{t^a}{t(t-1)\ldots(t-a+1)} \sum_{-r<n<-N} x^{(\alpha-a\beta)(n-u)} =$$

$$= \frac{a!t^a}{t(t-1)\ldots(t-a+1)} x^{-(\alpha-a\beta)u} \cdot \sum_{N<n<r} x^{(a\beta-\alpha)n} <$$

$$< \frac{a!t^a}{t(t-1)\ldots(t-a+1)} x^{(a\beta-\alpha)u} \frac{x^{(a\beta-\alpha)(N+1)}}{1-x^{a\beta-\alpha}} .$$

Furthermore we have

(56) $$\sum_{n>N} \frac{x^{\alpha(n-u)}}{1+(1+\frac{x^{\beta(n-u)}}{t})^t} < \sum_{n>N} x^{\alpha(n-u)} = x^{-\alpha u} \frac{x^{\alpha(N+1)}}{1-x^{\alpha}} .$$

Putting things together it follows that for all $N \in \mathbb{N}$

(57) $$\theta_x \leq a!\ x^{(a\beta-\alpha)u} . \frac{x^{(a\beta-\alpha)(N+1)}}{1-x^{a\beta-\alpha}} + \sum_{n=-N} \frac{x^{\alpha(n-u)}}{1+e^{x^{\beta(n-u)}}} + x^{-\alpha u} . \frac{x^{\alpha(N+1)}}{1-x^{\alpha}} .$$

Now let N tend to infinity in order to obtain

(58) $$\theta_x \leq \sum_{n=-\infty}^{\infty} \frac{x^{\alpha(n-u)}}{1+e^{x^{\beta(n-u)}}} .$$

Thus, combining (51) and (58) we find

(59) $$\theta_x = \sum_{n=-\infty}^{\infty} \frac{x^{\alpha(n-u)}}{1+e^{x^{\beta(n-u)}}} .$$

Similarly as before it follows that

(60) $$\theta_x = \sum_{n=-\infty}^{\infty} \frac{x^{-\alpha(n+u)}}{1+e^{x^{-\beta(n+u)}}} .$$

Changing the notation slightly we are thus led to the question whether there exist positive constants α and β such that the function $\theta(u)$ defined by

(61) $$\theta(u) = \sum_{n=-\infty}^{\infty} \frac{e^{\alpha(n+u)}}{1+e^{e^{\beta(n+u)}}} , \quad (u \in \mathbb{R})$$

is actually constant as a function of u.

We compute the Fourier coefficients a_ν, $\nu \in Z$, of $\theta(u)$. We have

(62) $$a_\nu = \frac{1}{2\pi}\int_0^{2\pi} e^{\nu ui}\theta(u)du = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_0^{2\pi} e^{\nu ui}\frac{e^{\alpha(k+\frac{u}{2\pi})}}{1+e^{e^{\beta(k+\frac{u}{2\pi})}}}du =$$

$$= \sum_{k=-\infty}^{\infty}\int_k^{k+1} e^{\nu(t-k)2\pi i}\frac{e^{\alpha t}}{1+e^{e^{\beta t}}}dt = \int_{-\infty}^{\infty} e^{\nu t2\pi i}\frac{e^{\alpha t}}{1+e^{e^{\beta t}}}dt =$$

$$= \int_0^{\infty} u^{\frac{\nu 2\pi i}{\beta}}\frac{u^{\frac{\alpha}{\beta}}}{1+e^u}\frac{du}{\beta u} = \frac{1}{\beta}\int_0^{\infty}\frac{u^{(\sigma+\frac{\nu 2\pi i}{\beta})-1}}{1+e^u}du = \frac{1}{\beta}\int_0^{\infty}\frac{u^{s-1}}{1+e^u}du = \frac{1}{\beta}\Gamma(s)\eta(s),$$

where $\sigma = \frac{\alpha}{\beta}$,

(63) $$s = s_\nu = \sigma + \frac{\nu 2\pi i}{\beta}$$

and

(64) $$\eta(s) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s} = (1-2^{1-s})\zeta(s), \quad (\mathrm{Re}\ s > 0).$$

Hence

(65) $$a_\nu = \frac{1}{\beta}\Gamma(s)(1-2^{1-s})\zeta(s).$$

CASE I: $\sigma > 1$.

Then $a_\nu \neq 0$ for all $\nu \in \mathbb{Z}$ because the Γ-function has no zeros at all, $\zeta(s) \neq 0$ on $\mathrm{Re}(s) > 1$ and also $1-2^{1-s} \neq 0$ for $\mathrm{Re}(s) > 1$ because all zeros of $1-2^{1-s}$ lie on the vertical line $\mathrm{Re}(s) = 1$. Hence, in case $\sigma > 1$, the periodic function under consideration is not constant for any choice of the positive parameters α and β.

CASE II: $\sigma = 1$.

Since it is known that $\zeta(s) \neq 0$ on the line $\mathrm{Re}(s) = 1$ we can have $a_\nu = 0$ only if $1-2^{1-s_\nu} = 0$. The zeros of $1-2^{1-s}$ are

(66) $$s = 1 + \frac{n2\pi i}{\log 2}, \quad n \in \mathbb{Z}.$$

Clearly $a_0 \neq 0$ so that we only have to find out when

(67) $$a_\nu = 1 + \frac{\nu 2\pi i}{\beta}$$

is one of the points

(68) $$1 + \frac{n2\pi i}{\log 2}$$

for all $\nu \in \mathbb{Z}\setminus\{0\}$.

This can only be the case when for any given $\nu \in \mathbb{Z}\setminus\{0\}$ there is an $n \in \mathbb{Z}$ such that

(69) $$\frac{\nu}{\beta} = \frac{n}{\log 2} .$$

Clearly β must be a rational multiple of log 2. Let $\beta = \frac{p}{q} \log 2$ where $p,q \in \mathbb{N}$ and $(p,q) = 1$. Then for every $\nu \in \mathbb{Z}\setminus\{0\}$ there must be an $n \in \mathbb{Z}$ such that

(70) $$\nu q = np.$$

Since $(p,q) = 1$ we must have $p|\nu$ for every $\nu \in \mathbb{Z}\setminus\{0\}$. Clearly this is only possible if $p = 1$. Hence $\beta = \frac{\log 2}{q}$. It follows that $a_\nu = 0$ for all $\nu \in \mathbb{Z}\setminus\{0\}$ if and only if $\beta = (\alpha=)\frac{\log 2}{q}$. More explicitly this means that the function

(71) $$\sum_{k=-\infty}^{\infty} \frac{e^{\alpha(k+u)}}{1+e^{e^{\alpha(k+u)}}}, \quad u \in \mathbb{R},$$

is actually a constant if and only if $\alpha = \frac{\log 2}{q}$ for some $q \in \mathbb{N}$. The case $\alpha = \log 2$ was discovered by B. VAN DER POL [WISKUNDIGE OPGAVEN, 19, I (1950) pp.308-311, Noordhoff N.V., Groningen].

CASE III: $0 < \sigma < 1$.

It will be clear now that in order to have $a_\nu = 0$, a_ν must be a zero of Riemann's ζ-function in the critical strip $0 < \sigma < 1$. Thus the function

$\theta(u)$ can only be constant if the set of nontrivial zeros of the ζ-function contains an arithmetical sequence of the form $\sigma + \frac{\nu 2\pi i}{\beta}$, $\nu \in \mathbb{N}$. Whether this is the case or not is clearly independent of the Riemann hypothesis. However, the problem involved seems to be most interesting in case $\sigma = \frac{1}{2}$. By means of the following argument we will only make *plausible* that the set of nontrivial zeros of $\zeta(s)$ does *not* contain a sequence as described above.

Assume that there are two positive numbers α and β such that $\sigma = \frac{\alpha}{\beta} < 1$ and

(72) $$\sum_{k=-\infty}^{\infty} \frac{e^{\alpha(k+u)}}{1+e^{e^{\beta(k+u)}}} = C, \ \forall u \in \mathbb{R},$$

for some constant C (depending on α and β). Writing $e^{\beta u} = t$ we obtain

(73) $$Ct^{-\sigma} = \sum_{k=-\infty}^{\infty} \frac{e^{\alpha k}}{1+e^{te^{\beta k}}}, \ \forall t > 0.$$

Taking Laplace transforms of both sides (which is feasible because of $\sigma < 1$) we find that

(74) $$C\Gamma(1-\sigma)s^{\sigma-1} = \sum_{k=-\infty}^{\infty} \int_0^{\infty} e^{-st} \frac{e^{\alpha k}}{1+e^{te^{\beta k}}} dt =$$

$$= \sum_{k=-\infty}^{\infty} e^{\alpha k} \int_0^{\infty} e^{-st}\Big(\sum_{m=1}^{\infty} (-1)^{m+1} e^{-mte^{\beta k}}\Big) dt =$$

$$\overset{(!)}{=} \sum_{k=-\infty}^{\infty} e^{\alpha k} \sum_{m=1}^{\infty} (-1)^{m+1} \int_0^{\infty} e^{-(s+me^{\beta k})t} dt =$$

$$= \sum_{k=-\infty}^{\infty} e^{\alpha k} \Big(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{s+me^{\beta k}}\Big), \ \forall s > 0.$$

Now observe that the function $s^{\sigma-1}$ can be continued analytically through the negative real axis whereas one gets the impression that the last mentioned series has no analytical continuation at all through the negative real axis.

However, we still lack a proof of this conjecture.

FINAL REMARK. It is still not known (to the author) whether the function $\theta_x(t)$ (defined by (49)) actually has a limit or not in case $\alpha = \beta = \frac{\log 2}{-q \log x}$ for some $q \in \mathbb{N}$. The only thing that can be said so far is that in case of existence

$$\lim_{t\to\infty} \theta_x(t) = \frac{\Gamma(1)\eta(1)}{\beta \log \frac{1}{x}} = q. \tag{75}$$